

Analysis of Tracking Performance of the MTDD Costas Loop for UQPSK Signal

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Even though the Costas loop for the Multimegabit Telemetry Demodulator/Detector (MTDD) System was originally designed for support of BPSK signalling, its inherent tracking capability for an unbalanced quadriphase shift-keyed (UQPSK) signal is well known. This paper summarizes analytic results to predict rms phase jitter of the MTDD Costas loop for a UQPSK signal. The Costas loop has a hard-limited in-phase channel.

I. Introduction

This paper summarizes the analytic results to predict the tracking performance of the breadboard Costas loop (Refs. 1, 2) for the Multimegabit Telemetry Demodulator/Detector (MTDD) System using an unbalanced quadriphase-shift-keyed (UQPSK) signal. The particular Costas loop is a biphasic polarity-type with passive arm filters. The loop contains a hard-limiter in front of the third multiplier, which is a chopper-type device. The merits of the polarity-type Costas loop have been well explained in Refs. 3 and 4.

Reference 4 has an excellent analysis on the polarity-type Costas loop tracking performance. However, the numerical results shown in the paper are not directly applicable to the prediction of the MTDD Costas loop at or in the vicinity of the design point SNR level. Basically, this paper extends the analysis of Ref. 4 to low input SNR cases.

The rms phase jitter predictions made in this paper have also been verified experimentally (Ref. 2).

II. Analysis

Since most of the analysis has been well documented in Ref. 4, only the minimum necessary equations will be summarized in this section for the sake of self-containment.

A. Loop Equation

The loop under consideration is shown in Fig. 1. Let the input signal be an unbalanced quadriphase-shift-keyed (UQPSK) signal.

$$s(t) = \sqrt{2P_2} m_2(t) \sin \Phi(t) + \sqrt{2P_1} m_1(t) \cos \Phi(t) \quad (1)$$

where

$$\Phi(t) \stackrel{\Delta}{=} \omega_0 t + \theta(t)$$

ω_0 = angular carrier frequency

$\theta(t)$ = received carrier phase

$$= \theta_0 + \Omega_0 t$$

θ_0 = a random phase

Ω_0 = a doppler frequency offset

$m_2(t), m_1(t)$ = binary data respectively for in-phase (I) and quadrature-phase (Q) channels

P_2, P_1 = average signal power respectively for I and Q channels.

The total received signal with additive noise is:

$$x(t) = s(t) + n_i(t) \quad (2)$$

where $n_i(t)$ is the additive bandpass channel noise which can be expressed in the following form (Ref. 5):

$$n_i(t) = \sqrt{2} \left\{ N_c(t) \cos \Phi(t) - N_s(t) \sin \Phi(t) \right\} \quad (3)$$

where $N_c(t)$ and $N_s(t)$ are approximately statistically independent, stationary, white Gaussian processes with single-sided noise spectral density N_0 (W/Hz).

Define the quadrature reference signals

$$\begin{aligned} r_s(t) &= \sqrt{2} K_1 \sin \hat{\Phi}(t) \\ r_c(t) &= \sqrt{2} K_1 \cos \hat{\Phi}(t) \end{aligned} \quad (4)$$

where K_1^2 is the rms power of the VCO output signal and $\hat{\Phi}(t)$ is the phase estimate of $\Phi(t)$.

Then, phase detector outputs are:

$$\begin{aligned} \epsilon_s(t) &\stackrel{\Delta}{=} K_m x(t) r_s(t) = K_1 K_m \left[\sqrt{P_2} m_2(t) - N_s(t) \right] \cos \phi(t) \\ &\quad - K_1 K_m \left[\sqrt{P_1} m_1(t) + N_c(t) \right] \sin \phi(t) \\ \epsilon_c(t) &\stackrel{\Delta}{=} K_m x(t) r_c(t) = K_1 K_m \left[\sqrt{P_2} m_2(t) - N_s(t) \right] \sin \phi(t) \\ &\quad + K_1 K_m \left[\sqrt{P_1} m_1(t) + N_c(t) \right] \cos \phi(t) \end{aligned} \quad (5)$$

where $\phi(t) = \Phi(t) - \hat{\Phi}(t) = \theta_0 - \hat{\theta}_0$ is the loop phase error, and K_m is the multiplier gain of the in-phase and quadrature-phase detectors. Then, the output signals of the in-phase and quadrature-phase arm filters with transfer function $G(s)$ are:

$$\begin{aligned} z_s(t) &\stackrel{\Delta}{=} G(p) \epsilon_s(t) = K_1 K_m \left\{ \left[\sqrt{P_2} \hat{m}_2(t) - \hat{N}_s(t) \right] \cos \phi(t) \right. \\ &\quad \left. - \left[\sqrt{P_1} \hat{m}_1(t) + \hat{N}_c(t) \right] \sin \phi(t) \right\} \\ z_c(t) &\stackrel{\Delta}{=} G(p) \epsilon_c(t) = K_1 K_m \left\{ \left[\sqrt{P_2} \hat{m}_2(t) - \hat{N}_s(t) \right] \sin \phi(t) \right. \\ &\quad \left. + \left[\sqrt{P_1} \hat{m}_1(t) + \hat{N}_c(t) \right] \cos \phi(t) \right\} \end{aligned} \quad (6)$$

where

$$\hat{m}_i(t) = G(p) m_i(t) \quad i = 1, 2$$

$$\hat{N}_\alpha(t) = G(p) N_\alpha(t) \quad \alpha = s, c$$

$G(p)$ = the Heaviside notation of the transfer function

In the derivation of Eq. (6), it is assumed that $\phi(t)$ is small, and unaffected by filtering. The output $z_0(t)$ of the chopper multiplier is given by product of $z_c(t)$ and the hard-limited $z_s(t)$:

$$\begin{aligned} z_0(t) &= z_c(t) \operatorname{sgn} [z_s(t)] = K_1 K_m \left\{ \sqrt{P_2} \hat{m}_2(t) \tilde{m}(t) \sin \phi(t) \right. \\ &\quad + \sqrt{P_1} \hat{m}_1(t) \tilde{m}(t) \cos \phi(t) - \hat{N}_s(t) \tilde{m}(t) \sin \phi(t) \\ &\quad \left. + \hat{N}_c(t) \tilde{m}(t) \cos \phi(t) \right\} \end{aligned} \quad (7)$$

where

$$\tilde{m}(t) \stackrel{\Delta}{=} \operatorname{sgn} [z_s(t)] ,$$

$$\operatorname{sgn}(x) = x/|x| .$$

The instantaneous frequency of the VCO output is related to $z_0(t)$ by

$$\frac{d\hat{\Phi}(t)}{dt} = K_v [F(p) z_0(t)] + \omega_0 \quad (8)$$

where K_v is the VCO gain in radians/volts. Then, the stochastic integro-differential equation of loop operation becomes

$$\begin{aligned} 2 \frac{d\phi(t)}{dt} &= 2\Omega_0 - K F(p) \left\{ 2 \sqrt{P_2} \hat{m}_2(t) \tilde{m}(t) \sin \phi(t) \right. \\ &\quad + 2 \sqrt{P_1} \hat{m}_1(t) \tilde{m}(t) \cos \phi(t) \\ &\quad \left. + 2 \tilde{m}(t) \hat{N}[t, \phi(t)] \right\} \end{aligned} \quad (9)$$

where $K \stackrel{\Delta}{=} K_1 K_m K_v$, and

$$\hat{N}[t, \phi(t)] \stackrel{\Delta}{=} \hat{N}_c(t) \cos \phi(t) - \hat{N}_s(t) \sin \phi(t) \quad (10)$$

As shown in Ref. 4, in the linear region of the loop the equation of loop operation can be approximately obtained:

$$2 \frac{d\phi}{dt} = 2\Omega_0 - K F(p) \left\{ \sqrt{P_2} \tilde{\alpha} f_1(2\phi) + \hat{N}_e(t, 2\phi) \right\} \quad (11)$$

where

$$\hat{N}_e(t, 2\phi) \stackrel{\Delta}{=} 2 \tilde{m}(t) \hat{N}(t, \phi) + n_\Delta(t, 2\phi) \quad (12)$$

n_Δ = self noise

$$= 2 \sqrt{P_1} \left[\hat{m}_1(t) \tilde{m}(t) - \langle \hat{m}_1(t) \tilde{m}(t) \rangle \right] \quad (13)$$

$\langle \rangle$ denotes the time average.

The overbar denotes the ensemble average

$f_1(x)$ = a nonlinearity which is periodic in x with period 2π and has unit slope at the origin

$\tilde{\alpha}$ = the signal suppression factor

$$= \frac{d}{d\phi} \left[\langle \hat{m}_2(t) \tilde{m}(t) \rangle \sin \phi(t) + \langle \hat{m}_1(t) \tilde{m}(t) \rangle \cos \phi(t) \right]_{\phi=0}$$

B. Signal Suppression Factor

Reference 4 first derived the signal suppression factor $\tilde{\alpha}$ for NRZ in-phase channel data and any quadrature channel binary data assuming single-pole (RC) Butterworth arm filters. The total suppression factor $\tilde{\alpha}$ can be expressed conveniently as the sum of α_2 and α_1 where α_2 is actually the suppression factor in the absence of the Q-channel and α_1 is a negative quantity which is a signal suppression due to the interference of the Q-channel. Thus,

$$\tilde{\alpha} = \alpha_2 + \alpha_1 \quad (14)$$

where

$$\alpha_2 = \frac{1}{2} \int_0^1 \left\{ 1 - 2 \exp[-2(B_i/R_2)x] \right\} \cdot \operatorname{erf} \left\{ \sqrt{\frac{\rho_2}{2}} \left\{ 1 - 2 \exp[-2(B_i/R_2)x] \right\} \right\} dx + \frac{1}{2} \operatorname{erf} \sqrt{\frac{\rho_2}{2}} \quad (15)$$

$$\alpha_1 = -\gamma_p \sqrt{\frac{2\rho_2}{\pi}} D_{m_1} \frac{1}{2} \int_0^1 \exp \left\{ -\frac{\rho_2}{2} \left\{ 1 - 2 \exp[-2(B_i/R_2)x] \right\}^2 \right\} dx - \gamma_p \sqrt{\frac{2\rho_2}{\pi}} D_{m_1} \frac{1}{2} \exp \left(-\frac{\rho_2}{2} \right) \quad (16)$$

$$\gamma_p = \frac{P_1}{P_2} = Q\text{-to-}I \text{ channel power ratio} \quad (17)$$

$$\rho_2 = \frac{2P_2}{N_0 B_l} \quad (18)$$

$$B_i = \frac{\omega_c}{2} = \text{two-sided arm filter input bandwidth} \quad (19)$$

ω_c = the 3-dB cutoff frequency of the arm filter with a transfer function

$$G(\omega) = \frac{1}{1 + j \frac{\omega}{\omega_c}} \quad (20)$$

R_2, R_1 = respectively I and Q channel data rates

D_{m_1} is the mean-squared power for the quadrature-channel data. For NRZ Q channel data,

$$D_{m_1} = 1 - \frac{1}{2 B_i/R_1} [1 - \exp(-2B_i/R_1)] \quad (21)$$

For a Manchester coded modulation in Q channel,

$$D_{m_1} = 1 - \frac{1}{2 B_i/R_1} [3 - 4 \exp(-B_i/R_1) + \exp(-2B_i/R_1)] \quad (22)$$

Using Eqs. (14) through (22), we can calculate the signal suppression factor by numerical integration.

C. Equivalent Noise Spectral Density for Small Input SNR

The equivalent noise spectral density is defined by:

$$N_e \triangleq 2 \int R_{\hat{N}_e}(\tau) d\tau \quad (23)$$

where

$$R_{\hat{N}_e}(\tau) \triangleq \langle \hat{N}(t, 2\phi) \hat{N}(t + \tau, 2\phi) \rangle \quad (24)$$

with straightforward derivation, Ref. 4 showed that

$$R_{\hat{N}_e}(\tau) = 8 \int_{-\infty}^{\infty} [R_{\hat{N}}(\tau) + P_1 R_{\hat{m}_1}(\tau)] R_y(\tau) d\tau \quad (25)$$

where

$$y(t) \stackrel{\Delta}{=} \text{sgn} \left\{ \sqrt{P_2} \hat{m}_2(t) - \hat{N}_s(t) \right\} \quad (26)$$

$$R_{\hat{N}}(\tau) = \frac{N_0 B_i}{2} e^{-2B_i |\tau|} \quad (27)$$

$R_{\hat{m}_1}(\tau)$ is given below

For a NRZ modulation (Ref. 3),

$$R_{m_1}(\tau) = \begin{cases} 1 - \frac{|\tau|}{T_1} + \frac{e^{-2B_i/R_1} \cosh 2B_i \tau - e^{-2B_i |\tau|}}{2B_i/R_1}; & 0 \leq |\tau| T_1 \\ \frac{e^{-2B_i |\tau|} [\cosh 2B_i/R_1 - 1]}{2B_i/R_1}; & T_1 \leq |\tau| \leq \infty \end{cases} \quad (28)$$

For a Manchester coded modulation (Ref. 4),

$$R_{\hat{m}_1}(\tau) = \begin{cases} 1 - 3 \frac{|\tau|}{T_1} + \frac{[4 e^{-B_i/R_1} - e^{-2B_i/R_1}] \cosh 2B_i \tau - 3 e^{-2B_i |\tau|}}{2B_i/R_1}; & 0 \leq |\tau| \leq \frac{T_1}{2} \\ - \left(1 - \frac{|\tau|}{T_1} \right) + \frac{e^{-2B_i |\tau|} [4 \cosh B_i/R_1 - 3] - e^{-2B_i/R_1} \cosh 2B_i \tau}{2B_i/R_1}; & \frac{T_1}{2} \leq |\tau| \leq T_1 \\ \frac{e^{-2B_i |\tau|} [4 \cosh B_i/R_1 - \cosh 2B_i/R_1 - 3]}{2B_i/R_1}; & T_1 \leq |\tau| \leq \infty \end{cases} \quad (29)$$

where $T_1 = 1/R_1$, $T_2 = 1/R_2$

Reference 3 presents approximate expressions for $R_y(\tau)$ corresponding to small ρ_2 and large ρ_2 cases.

Since we are concerned here with the low ρ_2 case, then we have:

$$R_y(\tau) = \frac{2}{\pi} \left[\sin^{-1} \rho_{\hat{N}}(\tau) - \rho_2 \frac{R_{\hat{m}_2}(0) \rho_{\hat{N}}(\tau)}{\sqrt{1 - \rho_{\hat{N}}^2(\tau)}} + \rho_2 \frac{R_{\hat{m}_2}(\tau)}{\sqrt{1 - \rho_{\hat{N}}^2(\tau)}} \right] \quad (30)$$

where

$$\rho_{\hat{N}}(\tau) = e^{-2B_1 \tau}$$

$$R_{\hat{m}_2}(\tau) = \begin{cases} 1 - \frac{|\tau|}{T_2} + \frac{e^{-2B_i/R_2} \cosh 2B_i \tau - e^{-2B_i|\tau|}}{2B_i/R_2}; & 0 \leq |\tau| \leq T_2 \\ \frac{e^{-2B_i|\tau|} [\cosh 2B_i/R_2 - 1]}{2B_i/R_2}; & T_2 \leq |\tau| < \infty \end{cases} \quad (31)$$

$$R_{\hat{m}_2}(0) = 1 - \frac{1 - e^{-2B_i/R_2}}{2B_i/R_2} \quad (32)$$

Now we have all the equations needed to compute the equivalent noise spectral density. Since an analytic integration of the integral in Eq. (25) is cumbersome, a numerical integration may be attempted. However, $R_y(\tau)$ in Eq. (30) contains terms which go to infinity when $\tau = 0$. Thus, a change of forms of equation is required.

First let's approximate the integration region $0 \leq |\tau| \leq 2T_2$. Also let's consider β defined by:

$$\beta = \frac{N_e}{4 N_0} \quad (33)$$

Then,

$$\beta = \frac{2}{N_0} \int_{-2T_2}^{2T_2} \left[R_{\hat{N}}(\tau) + P_1 R_{\hat{m}_1}(\tau) \right] R_y(\tau) d\tau \quad (34)$$

As in the calculation of the signal suppression factor, the total equivalent noise factor β may be expressed as the sum of two factors β_1 and β_2 :

$$\beta = \beta_2 + \beta_1$$

where

$$\beta_2 = \frac{2}{N_0} \int_{-2T_2}^{2T_2} R_{\hat{N}}(\tau) R_y(\tau) d\tau \quad (35)$$

$$\beta_1 = \frac{2 P_1}{N_0} \int_{-2T_2}^{2T_2} R_{\hat{m}_1}(\tau) R_y(\tau) d\tau \quad (36)$$

It is observed that β_2 is the noise factor in the absence of Q channel and β_1 is the increase of noise due to the presence of Q channel data.

By change of variable $2T_2x = \tau$, we have

$$\beta_2 = 4 \frac{B_i}{R_2} \int_0^1 \exp(-4B_i/R_2) R_y \left(\frac{2x}{R_2} \right) dx \quad (37)$$

$$\beta_1 = 4 \rho_1 \frac{B_i}{R_2} \int_0^1 R_{\hat{m}_1} \left(\frac{2x}{R_2} \right) R_y \left(\frac{2x}{R_2} \right) dx \quad (38)$$

where

$$\rho_1 = \frac{2P_1}{N_0 B_i} \quad (39)$$

$$R_{\hat{m}_1} \left(\frac{2x}{R_2} \right) = 1 - 2x \frac{R_1}{R_2} + \frac{\exp(-2B_i/R_1) \cosh \left(4 \frac{B_i}{R_2} x \right) - g(x)}{2 \frac{B_i}{R_1}} \quad (40)$$

$$g(x) = \exp \left(-4 \frac{B_i}{R_2} x \right) \quad (41)$$

For $0 \leq x < 0.5$,

$$R_y \left(\frac{2x}{R_2} \right) = \frac{2}{\pi} \left[\sin^{-1} g(x) + \rho_2 \left(\frac{\sqrt{1-g(x)}}{\sqrt{1+g(x)}} + \frac{\exp(-2B_i/R_2) \sqrt{1-g^2(x)}}{4B_i/R_2 g(x)} \right) - \rho_2 \cdot 2x \cdot \frac{1}{\sqrt{1+g(x)} \sqrt{h(x)}} \right] \quad (42)$$

$$h(x) = 1 - g(x) = ax \left(1 - \frac{ax}{2} + \frac{(ax)^2}{3!} - \frac{(ax)^3}{4!} + \frac{(ax)^4}{5!} \dots \right) \quad (43)$$

$$a = 4B_i/R_2 \quad (44)$$

Equation (42) is obtained from (30) and (31) after straightforward derivation. Taylor series expansion of $g(x)$ is used to prevent an overflow in a numerical integration at the vicinity of $x = 0$. For $0.5 < x \leq 1$, a straightforward substitution in Eq. (30), (31) and (32) gives

$$R_y \left(\frac{2x}{R_2} \right), 0.5 < x \leq 1.0.$$

D. RMS Phase Jitter

In a linear region of the loop, the rms phase jitter of ϕ is given (Ref. 4) by

$$\sigma_{\phi}^2 = \frac{1}{\rho S_L} \quad (45)$$

$$\rho = \frac{P}{N_0 B_L} \quad (46)$$

where

$$P = P_1 + P_2$$

$$B_L = \text{single-sided loop bandwidth}$$

$$N_0 = \text{single-sided noise spectral density}$$

$$S_L = \text{squaring loss}$$

$$= \frac{1}{(1 + \gamma_p)} \frac{\alpha^2}{\beta} \quad (47)$$

$$\gamma_p = P_1/P_2$$

We already have obtained all necessary equations to calculate the squaring loss S_L . Thus the only thing left is to express the loop bandwidth B_L as a function of input SNR.

The loop filter of the Costas loop under consideration is of the imperfect second-order loop type with the following transfer function:

$$F(s) = \frac{1 + \tau_2 s}{1 + \tau_1 s} \quad (48)$$

The bandwidth of the loop is shown in Ref. 6 to be

$$B_L = \frac{\omega_n}{2} \left(\zeta + \frac{1}{4\zeta} \right) \quad (49)$$

where

$$\omega_n = \sqrt{\frac{K}{\tau_1}} = \text{natural angular frequency}$$

$$\zeta = \frac{1 + K \tau_2}{2 \omega_n \tau_1} = \text{loop damping factor}$$

$$K = \text{total loop gain}$$

Suppose K_0 is the loop gain at the design point, i.e., at $PT_2/N_0 = -4$ dB (Ref. 1). Then, the loop gain at a arbitrary SNR is given by:

$$K = K_0 \cdot \frac{\tilde{\alpha}}{\tilde{\alpha}_0} \cdot \frac{g\sqrt{P}}{(g_0\sqrt{P})} \quad (50)$$

where $\tilde{\alpha}_0, \tilde{\alpha}$ are the signal suppression factors respectively at the design point and a desired SNR, g_0, g are the AGC or MGC IF gain control factor at the design point or a desired SNR.

Using (49) and (50), we can obtain the loop bandwidth at any SNR level. Thus we are now ready to calculate the rms phase jitter at any SNR level.

III. Numerical Results and Discussions

Figures 2-9 show the plots of squaring loss for various conditions. Figure 10 shows the rms phase jitter for the MTDD breadboard Costas loop with various conditions (see also Ref. 7). The rms phase jitter is a function of PT_2/N_0 , γ_p and R_2/R_1 . Obviously, the phase jitter will decrease as PT_2/N_0 increases, γ_p decreases or R_2/R_1 increases.

Generally, for low input SNR or PT_2/N_0 in the range from -4 to $+2$ dB, the squaring loss is an increasing function of PT_2/N_0 . At higher PT_2/N_0 the trend reverses (Ref. 4). However, as mentioned above, the rms phase jitter is a monotonically decreasing function of PT_2/N_0 .

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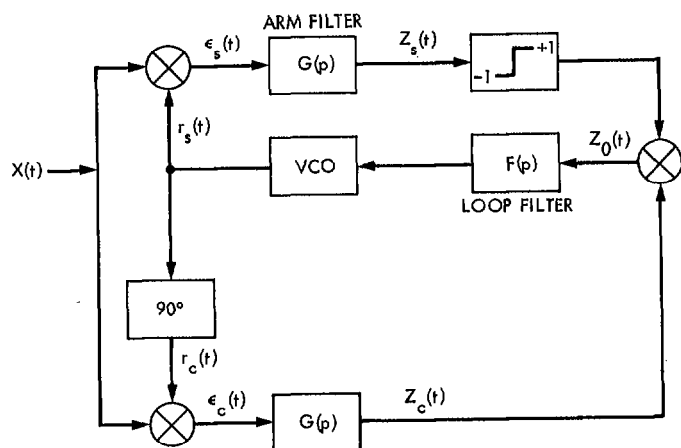


Fig. 1. Costas loop block diagram

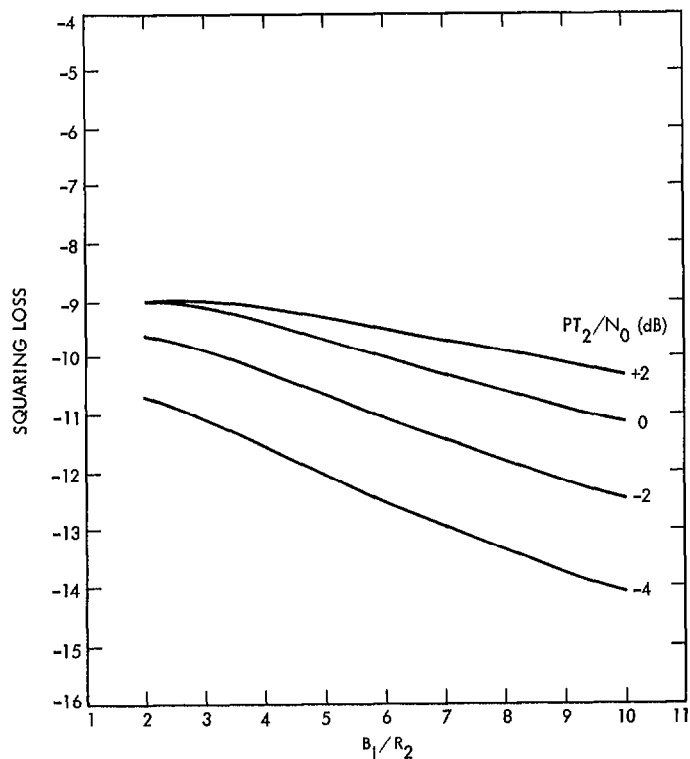


Fig. 3. Squaring loss ($R_2/R_1 = 5.0$; $\gamma_p = 0.25$)

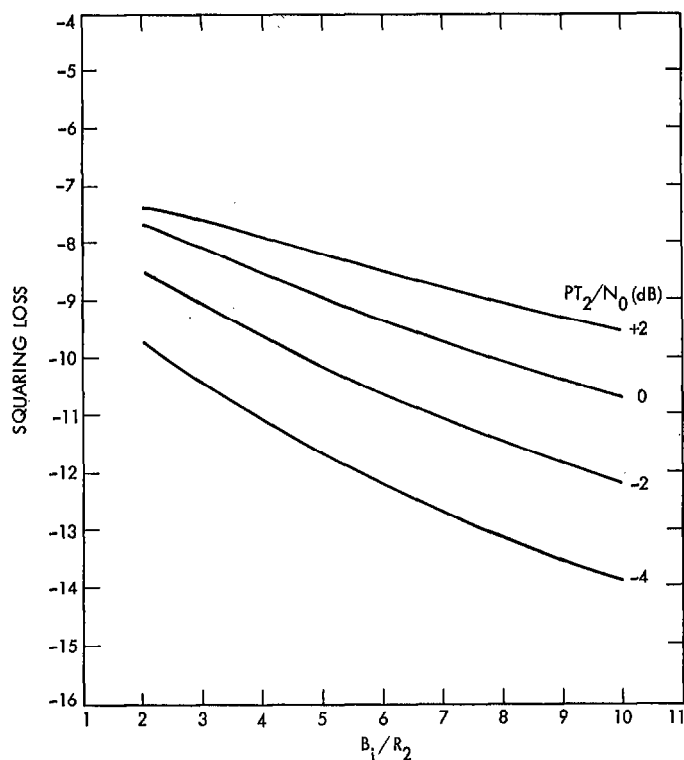


Fig. 2. Squaring loss ($R_2/R_1 = 1.0$; $\gamma_p = 0.25$)

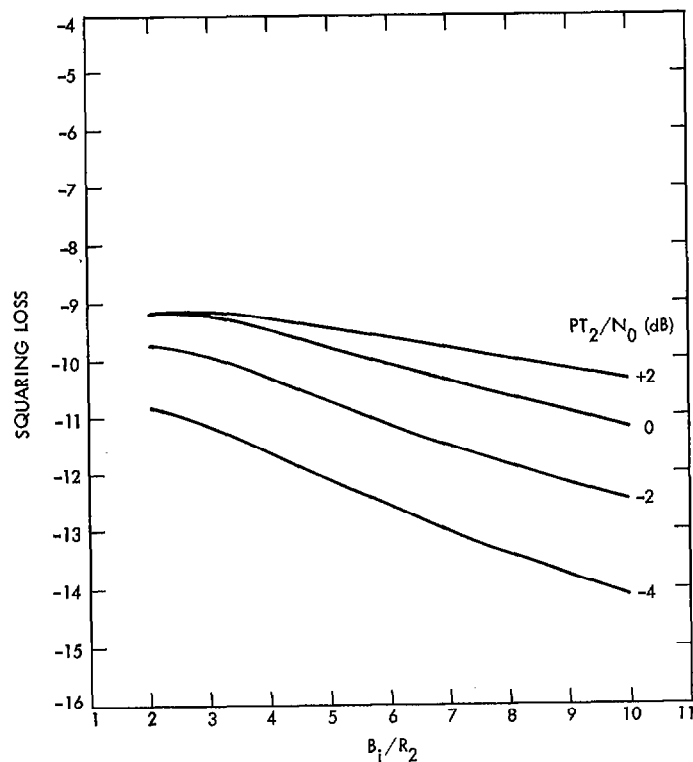
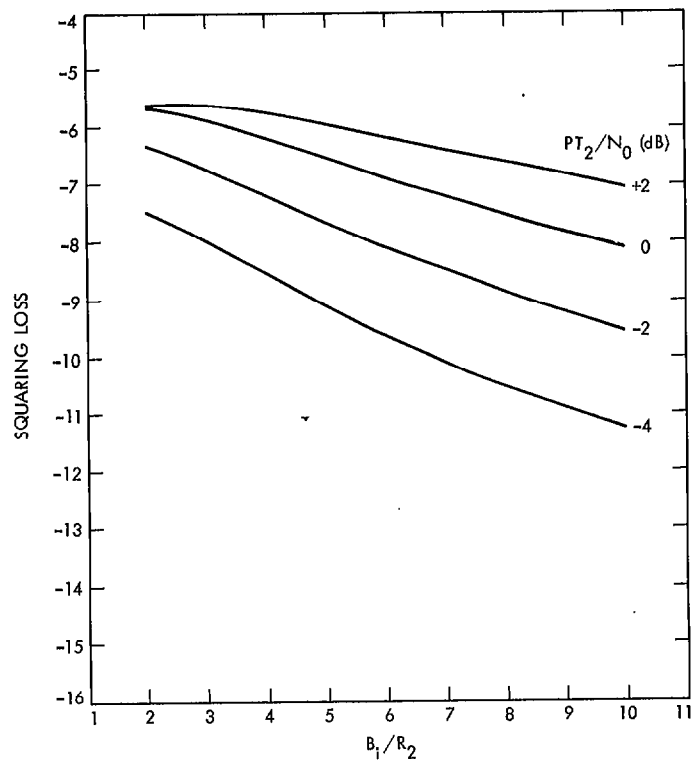
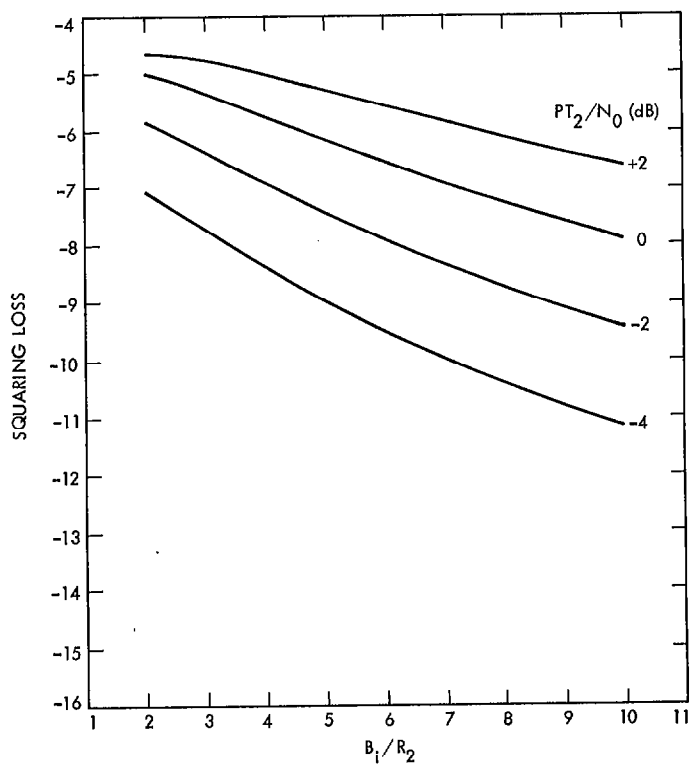
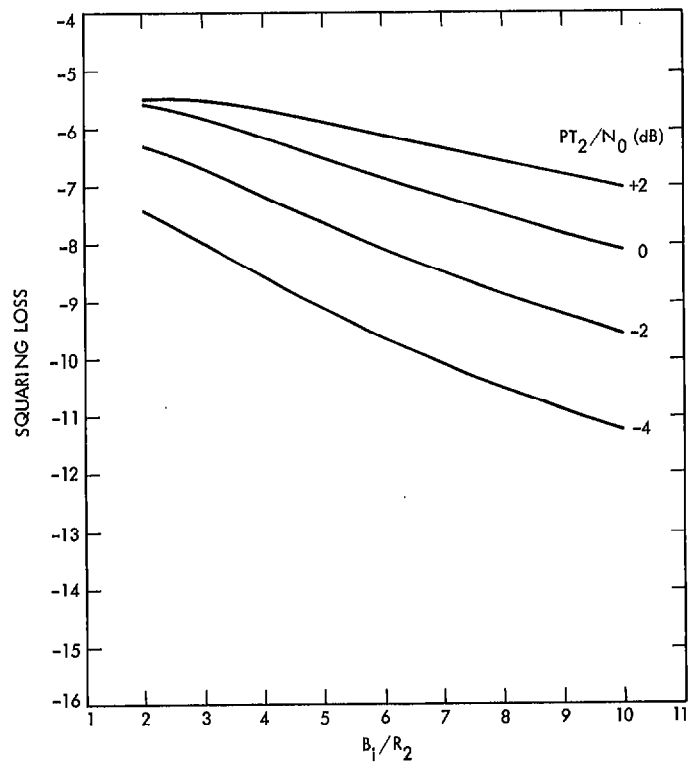
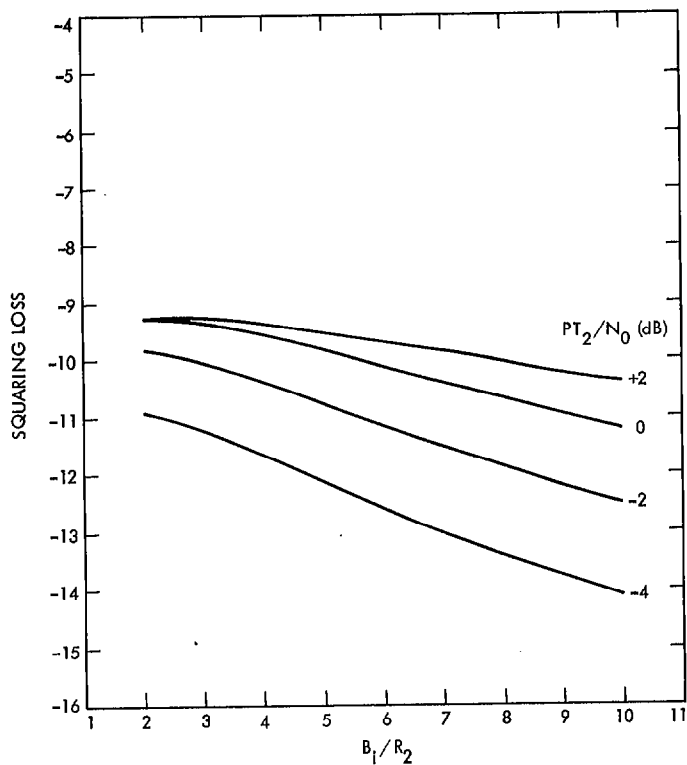


Fig. 4. Squaring loss ($R_2/R_1 = 10$; $\gamma_p = 0.25$)



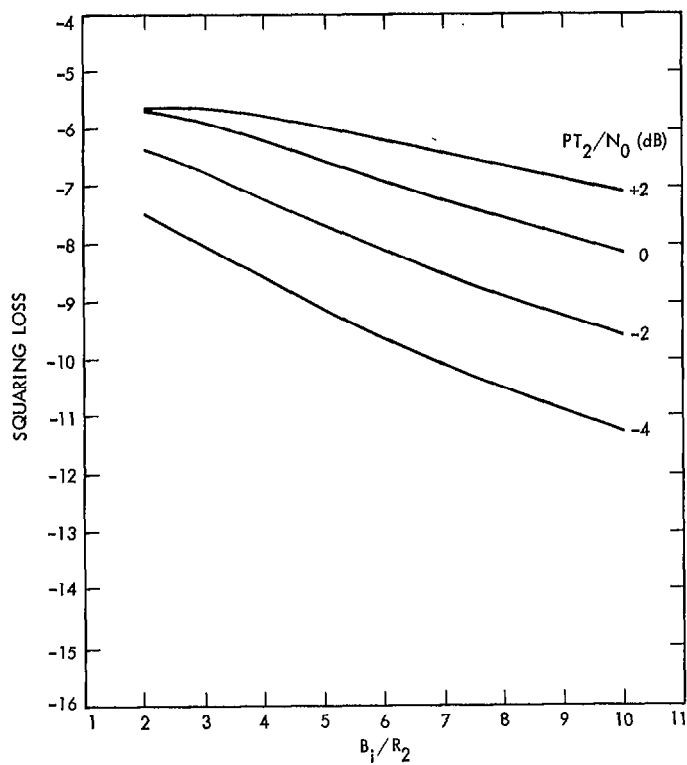


Fig. 9. Squaring loss ($R_2/R_1 = 20$; $\gamma_p = 0.1$)

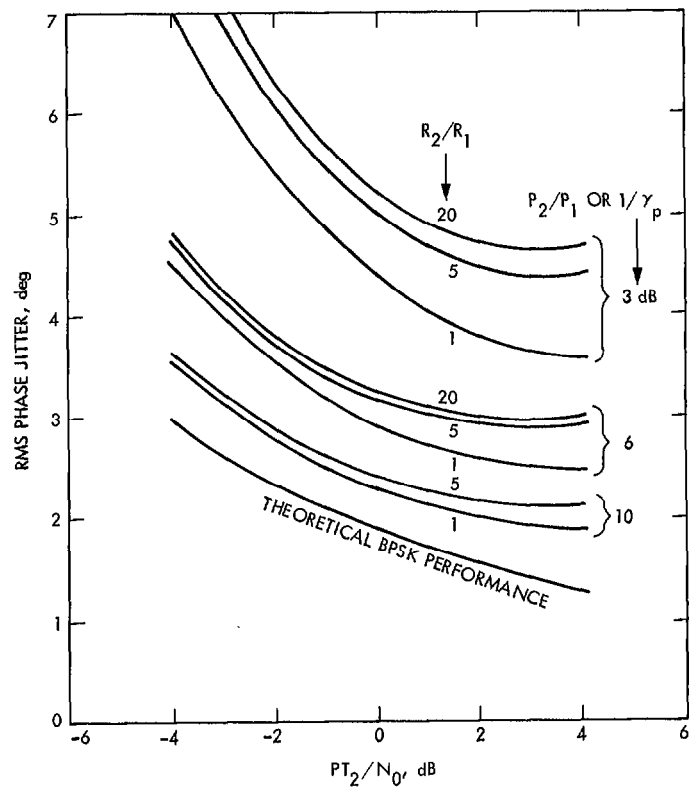


Fig. 10. Theoretical rms phase jitter calculated with low input SNR approximation for MTDD breadboard Costas loop using UQPSK signals